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Non-crystallographic nets: characterization and first steps towards a classification[1]

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Non-crystallographic (NC) nets are periodic nets characterized by the existence of non-trivial bounded automorphisms. Such automorphisms cannot be associated with any crystallographic symmetry in realizations of the net by crystal structures. It is shown that bounded automorphisms of finite order form a normal subgroup $F(N)$ of the automorphism group of NC nets (N, T). As a consequence, NC nets are unstable nets (they display vertex collisions in any barycentric representation) and, conversely, stable nets are crystallographic nets. The labelled quotient graphs of NC nets are characterized by the existence of an equivoltage partition (a partition of the vertex set that preserves label vectors over edges between cells). A classification of NC nets is proposed on the basis of (i) their relationship to the crystallographic net with a homeomorphic barycentric representation and (ii) the structure of the subgroup $F(N)$.

1. Introduction

Since the precursor works of Wells (1977) and O'Keeffe & Hyde (1980), the topology of crystal structures has been represented by periodic nets. Whereas such nets, as combinatorial objects, may display a rich variety of automorphism groups, the emphasis has naturally been put on periodic nets whose automorphism group is isomorphic to some space group. These have been called crystallographic nets (Klee, 2004). By contrast, periodic nets that do not satisfy this restriction have been called non-crystallographic (NC) nets.

Relatively few NC nets are known, to which observation one may assign two extreme explanations. For some reason it might happen that crystal structures with the topology of NC nets are not favoured from a thermodynamic, or maybe a kinetic, point of view, and are consequently rare in nature. But it might also be that, in the absence of easily identifiable features, one fails to recognize them.

The easiest way to deal with periodic nets (N, T) is to work with their labelled quotient graphs N/T, informally the graph of their vertex- and edge-lattices (Chung *et al.*, 1984). However, only automorphisms in the normalizer of the translation group T – called periodic automorphisms in Delgado-Friedrichs (2005) – induce an automorphism of N/T. When present, such automorphisms of N/T are easily detected, but this is not the rule. In general, one must still rely on a direct study of periodic nets to analyse their full automorphism group.

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Consider, for example, the labelled (*voltage*) graph, say Q, shown in Fig. 1(*a*). By the definition of voltage graphs the derived 2-periodic net N contains the three vertex-lattices $A = \{A_{i,j}\}$, $B = \{B_{i,j}\}$ and $C = \{C_{i,j}\}$ for $i, j \in \mathbb{Z}$. Edge-lattices are derived in the same way. For example, edge AB in Q with label vector (voltage) 10 generates the edge-lattice $A_{i,j}B_{i+1,j+0}$. The derived net is shown in Fig. 1(*b*) where the same colour has been used for an edge of Q and the respective edge-lattice. The following analysis of the automorphisms of Q shows that the normalizer of the translation group $T = \mathbb{Z}^2$ is isomorphic to *p*2*mm* leading to the representation of the derived 2-periodic net in this space group in Fig. 1(*c*). Indeed, a first generator μ_1 of the automorphism group of Q reverses the loop at vertex A and exchanges the two parallel edges BC with voltage 01 while leaving invariant edges with voltage 10. A second generator μ_2 exchanges vertices B and C, as well as edges AB and AC, exchanges and simultaneously reverses the two edges BC with voltage 01 while it reverses edge BC with voltage 10 and fixes the loop at A. It may be seen that μ_1 reverses direction 01 and fixes direction 10 of the lattice while μ_2 reverses direction 10 and fixes direction 01: together these operations generate the point group 2*mm* [see Klee (2004) or Eon (2011) for a step-by-step description of the method]. Unexpectedly, however, the net is uninodal and non-crystallographic: the permutation $\pi = \prod_{i,j}(A_{i,2j}, B_{i,2j}, C_{i,2j})(A_{i,2j+1}, C_{i,2j+1}, B_{i,2j+1})$ for $i, j \in \mathbb{Z}$, which exchanges the three vertices in every unit cell, but in a different way according to whether the index j of the unit cell is odd or even, is indeed non-crystallographic. Because it does not respect the lattices it cannot induce an automorphism in Q and it cannot be associated with an isometry in a Euclidean

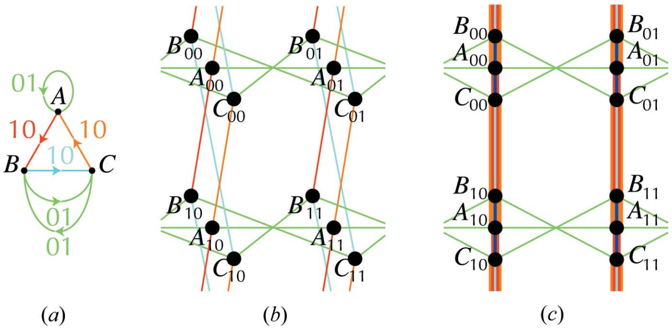

Figure 1
(*a*) A labelled graph, (*b*) a representation of the derived 2-periodic net (the triple-**sql**) with crossings and (*c*) a representation of this net in space group *p*2*mm* with points in Wyckoff positions *a* (site symmetry 2*mm*: point-lattice *A*) and *e* (site symmetry ..*m*: point-lattices *B* and *C*): line-lattices have the same hue in the net and its quotient. Different line-lattices also have different width in (*c*) to help in distinguishing superimposed lines along direction 10; light-blue lines $B_{i,j}C_{i+1,j}$ and $B_{i+1,j}C_{i+2,j}$, from the same line-lattice *BC* are also superimposed: the common segments $B_{i+1,j}C_{i+1,j}$ have been shaded.

representation of the net unless the three vertex-lattices are represented by the same point-lattice: it is then associated with the null-vector translation since it has order 3. This automorphism is an example of a *bounded automorphism* in the sense that the graph-theoretical distance between any vertex and its image by π is uniformly bounded. Non-translational bounded automorphisms are characteristic of NC nets (Eon, 2005). Moreover, π generates the group $F(N) = \{1, \pi, \pi^2\}$ of bounded automorphisms of finite order. The orbits by π in the vertex set of N are exactly the subsets $\{A_{i,j}, B_{i,j}, C_{i,j}\}$ that partition the vertex set of the net into the so-called *blocks of imprimitivity*, *i.e.* a partition which is preserved by any automorphism of the net.

Some typical examples of NC nets were studied in the two previous papers of the authors. An analysis of NC nets with freely acting (*i.e.* with no fixed vertices) bounded automorphism groups was reported in Moriera de Oliveira Jr & Eon (2011). The net drawn in Fig. 1 is perhaps the simplest net in this class and was thoroughly analysed in that paper. More recently, Moriera de Oliveira Jr & Eon (2013) put the emphasis on NC nets (N, T) admitting a system of finite blocks of imprimitivity for the group $B(N)$ of bounded automorphisms. Nets in both classes share some remarkable properties.

In both cases, NC nets are unstable, that is, any barycentric representation displays vertex collisions. Moreover, provided origins in vertex-lattices are suitably chosen, it is possible to evidence an *equivoltage partition* (a partition of the vertex set into disjoint *cells* – or parts – such that voltages over edges between two cells are preserved) in the labelled quotient graph of these nets. For example, the quotient graph Q in Fig. 1(*a*) admits an equivoltage partition with a single cell: every vertex in this graph is the origin and the end of edges with voltages 10 and 01.

Bounded automorphisms of finite order also appear to play a central role in both classes of NC nets, as they were observed to form a non-trivial subgroup $F(N) < B(N)$. In this article, we show that the latter property is quite general and characterizes the whole class of NC nets. In other words, in any NC net (N, T), the set of bounded automorphisms of finite order is stable under composition and constitutes a normal subgroup of $B(N)$, and indeed of the full group $\mathrm{Aut}(N)$ of automorphisms of the net. This result ensures the existence of a system of finite blocks of imprimitivity for $B(N)$ and, consequently, the existence of an equivoltage partition of the labelled quotient graph N/T. Hence, equivoltage partitions in labelled quotient graphs appear to be the fingerprint of NC nets.

It has already been stated in Delgado-Friedrichs (2005) that NC nets are unstable nets. One should consider, however, that only periodic automorphisms were taken into account in that work. In other words, those automorphisms of the net that do not respect its vertex-lattices were excluded *a priori* from the definition of the automorphism group of a periodic net, denoted as $\mathrm{Aut}(N, T)$. Here, we consider the full group of automorphisms $\mathrm{Aut}(N)$ of the net. We emphasize that, as shown in our previous example, the group of periodic automorphisms $\mathrm{Aut}(N, T)$ of a periodic net (N, T) may be isomorphic to some space group while the full group of automorphisms $\mathrm{Aut}(N)$ is not. In this sense, our results using the wider definition of automorphisms for periodic nets extend the conclusions reached in the above-mentioned work: NC nets are unstable nets and conversely, but maybe most importantly, stable nets are crystallographic nets. Moreover, labelled quotient graphs remain the best tool to perform the analysis. Using the narrower definition, periodic automorphisms of the net are associated with those automorphisms of the labelled quotient graph that are consistent with the voltages over its cycles. Using the wider definition of automorphisms, the quotient graph of NC nets should display an equivoltage partition; given a labelled quotient graph, the concept of equivoltage partition is patently a generalization of that of automorphism.

Our analysis makes wide use of geodesic fibres, defined as minimal 1-periodic subgraphs of the net and introduced in Eon (2007). Geodesic fibres in the 2-periodic net shown in Fig. 1(*b*) are infinite paths such as $\ldots, A_{i,j}, B_{i+1,j}, C_{i+2,j}, A_{i+3,j}, \ldots$. We show that geodesic fibres of a p-periodic network along a given direction may be built into a $(p-1)$-periodic network. The term *network* is used here as a weakened substitute for *net*, when the 3-connectedness condition is released to simple connectedness. The descent in network periodicity is the basis for an inductive proof of the cornerstone result concerning the stability of the set of bounded automorphisms of finite order under composition.

The paper is organized as follows. Basic concepts on periodic nets (and networks) and their geodesic fibres are summarized in §§2 and 3. As a useful application, it is shown in §4 that lattice nets are crystallographic nets. Fibre networks are constructed and their properties analysed in §5. The stability of the set of bounded automorphisms of finite order under composition is presented in the next two sections. An analysis of 1-periodic networks in §6 is the first step of an argument by induction on the periodicity of networks, which is given in §7. The fundamental results characterizing NC nets and leading to a classification scheme are derived in §8. Two simple examples are described in §§9 and 10.

2. Preliminary concepts

A *net* has been defined by Klee (2004) as a simple, locally finite, 3-connected graph. By definition, *simple* graphs have no loops and no multiple edges; vertices in *locally finite* graphs have a finite number of neighbours. These two conditions are clearly necessary if the graph is to represent the bonds in a chemical compound. The connectivity required here is the *point-connectivity* and corresponds to the smallest number of vertices that should be withdrawn to disconnect the graph. Note that deleting a vertex also requires deleting all incident edges. Because the 3-connectivity condition is not always satisfied (nor required) in this work, we define in parallel a *network* as a simple, locally finite, connected graph. For example, the *ladder* in Fig. 2(*a*) is 2-connected, the *double ladder* in Fig. 2(*b*) is 3-connected and the square lattice net **sql** in Fig. 2(*c*) is 4-connected: the ladder is then a network while the double ladder and **sql** are nets. Observe that the ladder is 2-connected even though all its vertices have degree 3.

Two vertices linked by an edge are said to be *adjacent*. *Automorphisms* of a net are vertex permutations that preserve the adjacency relationship; the automorphism group of a net N is denoted by $\mathrm{Aut}(N)$.

A *p-periodic* net (or network) (N, T) is constituted of a net (or network) N and a free abelian group $T \leq \mathrm{Aut}(N)$ of rank p, such that the number of vertex and edge orbits by T (called *vertex-lattices* and *edge-lattices*, respectively) in N is finite. Periodic nets (N, T) such that the automorphism group $\mathrm{Aut}(N)$ is not isomorphic to any isometry group in the Euclidean space are called *non-crystallographic nets* and are characterized by the presence of non-translational *bounded automorphisms*, *i.e.* automorphisms $\varphi \notin T$ such that the distance between a vertex and its image is uniformly bounded (Eon, 2005).

A system σ of finite blocks of imprimitivity for the group of bounded automorphisms $B(N)$ is a partition σ into finite cells (called blocks) of the vertex set of the periodic net (N, T) which is preserved by bounded automorphisms. That is, if $\Delta \in \sigma$ and $\varphi \in B(N)$, then $\varphi(\Delta) \in \sigma$.

The *quotient graph* of the periodic net (or network) (N, T) is the graph N/T of its vertex- and edge-lattices. A *labelled quotient graph* is obtained after assigning lattice vectors to the edges of the quotient graph. This requires setting an origin for each vertex-lattice in the net and choosing an orientation for the edges of the quotient graph; the label vector (voltage) assigned to some edge $e = AB$ in N/T is the p-tuple t of the edge A_0B_t from the corresponding edge-lattice in N.

A partition of a graph into cells C_i is *equitable* if the number of edges with end-vertex in the cell C_j starting from a given vertex V in cell C_i is a number b_{ij}, independent of the chosen vertex in C_i. We say that a partition of a labelled quotient graph is *equivoltage* if it is equitable and if the *multiset* (*i.e.* the set including multiplicity) of b_{ij} voltages over the edges going from some vertex V in C_i to vertices in C_j is also independent of V (see Moreira de Oliveira Jr & Eon, 2013). Illustrations of these concepts are provided in Fig. 3. It should be observed that equivoltage partitions may exist even in the absence of automorphisms preserving voltages.

Barycentric representations of periodic nets play an important role in the study of NC nets. A Euclidean representation of the net is a mapping ρ of the vertex and edge sets to points and line segments, respectively, in Euclidean space such that $\rho(e) = \rho(A)\rho(B)$ for the edge $e = AB$. A representation is periodic if any vertex-lattice and edge-lattice are mapped on a point-lattice and a line-lattice, respectively. In barycentric representations, the image of every vertex is at the centre of gravity of the images of its first neighbours. Because periodic, barycentric representations of NC nets display vertex collisions their determination must follow the cycle–cocycle method described in Eon (2011). We recall below a property of bounded automorphisms with fixed vertices that was derived in Moreira de Oliveira Jr & Eon (2013).

Corollary 2.1. Suppose there is a non-trivial automorphism f of a periodic net (N, T) that fixes every vertex in some vertex-lattice $[X]$. Then any periodic, barycentric representation of

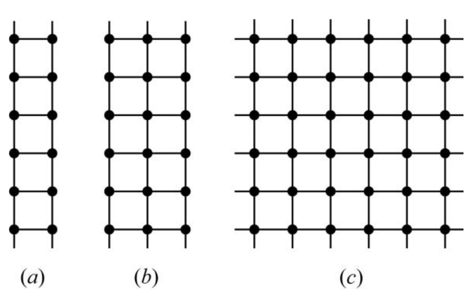

Figure 2
(*a*) The ladder, (*b*) the double ladder and (*c*) the square lattice net **sql**.

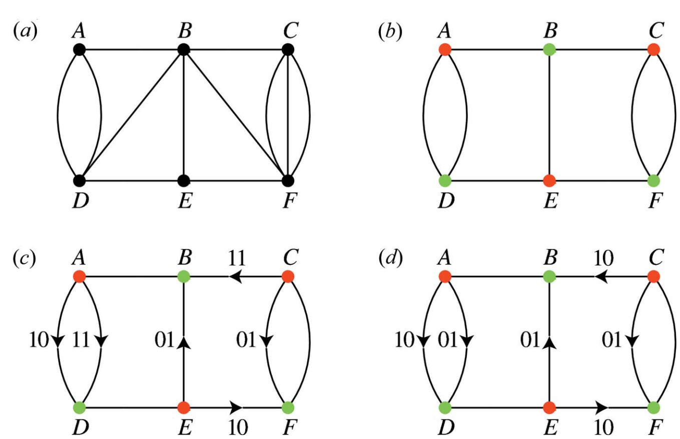

Figure 3
(*a*) A graph with no non-trivial equitable partition, (*b*) a graph with an equitable partition (every red vertex makes three links to green vertices and every green vertex makes three links to red vertices), (*c*) a voltage graph with no non-trivial equivoltage partition and (*d*) a voltage graph with an equivoltage partition (red vertices show the same voltage multisets, {10, 01, 00} for edges linking them to the green ones and $\{\bar{1}0, 0\bar{1}, 00\}$ for the edges from green to red vertices).

the net in Euclidean space presents vertex collisions. In particular, every vertex in (N, T) is mapped on the same point as its image by *f*.

3. Geodesic fibres

This section summarizes the main definitions and results concerning geodesic fibres in periodic nets (see Eon, 2007).

Given a subgroup $S < T$ of rank 1, we denote by $\mathrm{Ext}(S)$ the maximal extension (*i.e.* the largest supergroup) of S of rank 1 such that $S \le \mathrm{Ext}(S) \le T$. For instance, if $S = \langle(4, 8)\rangle$ is the subgroup generated in $\mathbb{Z}^2$ by the translation $(4, 8)$, then $\mathrm{Ext}(S) = \langle(1, 2)\rangle$ is the subgroup generated by the translation $(1, 2)$. Of course, the index of S in $\mathrm{Ext}(S)$ is finite.

We recall that a subgraph F of a graph G is said to be *geodesically complete* in G if, for any pair of vertices $U, V \in G$, F contains all geodesic (shortest) paths UV in G.

Definition 3.1. A periodic subgraph (F, S) of a periodic graph (G, T) is called a geodesic T-fibre or simply a fibre if (*a*) the translation group S is a subgroup of T of rank 1, (*b*) the subgraph F is geodesically complete in G and (*c*) F is minimal with respect to the conditions of periodicity (*a*) and completeness (*b*). We say that the fibre (F, S) is along S. Two T-fibres (F_1, S_1) and (F_2, S_2) such that $\mathrm{Ext}(S_1) = \mathrm{Ext}(S_2)$ are said to be parallel.

Fig. 4 illustrates the restrictions imposed by the above definition. Here we consider the honeycomb net **hcb** and some 1-periodic subgraphs (networks) that have been highlighted in red for clarity. Fig. 4(*a*) shows a 1-periodic subgraph with translation group generated by the translation 22 of **hcb**, which is topologically equivalent (*homeomorphic*) to the ladder in Fig. 2(*a*). This network is geodesically incomplete since the missing *rungs* are shortcuts in **hcb** to paths in the network between the respective end-vertices. Adding these edges, as in Fig. 4(*b*), one gets a 1-periodic subgraph that is geodesically complete: every shortest path in **hcb** between any pair of vertices in this network belongs to it but it is not minimal. It contains as a subgraph a 1-periodic network isomorphic to the infinite path shown in Fig. 4(*c*). The latter is geodesically complete and minimal, since withdrawing any edge turns it geodesically incomplete.

Examples of fibres in 2-periodic nets are given in Fig. 5. The net 4.8^2 shown in Fig. 5(*a*) has fibres that are infinite paths along $1\overline{1}$ (green) and non-trivial fibres along 10 (red). Similarly the net $4.3.4.3^2$ in Fig. 5(*b*) admits fibres that are infinite paths along 01 (green) and non-trivial fibres along 11 (red). Examples of fibres in 3-periodic nets will be described in §5.

The main properties of geodesic fibres and those related to bounded automorphisms were disclosed in Eon (2007); they are enumerated below:

(i) Any p-periodic net (N, T) admits geodesic fibres along at least p independent directions.

(ii) The quotient graph of a fibre F/S is a subgraph of the quotient graph N/T of the net.

(iii) In any periodic net, bounded automorphisms map geodesic fibres to parallel geodesic fibres.

(iv) Bounded automorphisms with fixed vertices have finite order.

Property (ii) permitted us to draw the labelled quotient graphs of the geodesic fibres displayed in Fig. 5 as labelled subgraphs of the quotient graph of the respective nets, *i.e.* the labelled quotient graph of the fibre is obtained from that of the net by deleting some of the edges. Property (iii) makes fibres the fundamental tool for studying bounded automorphisms in NC nets.

4. Lattice nets

Periodic nets with a single vertex-lattice have been called *lattice nets* (Delgado-Friedrichs & O'Keeffe, 2009). Their quotient graph contains a single vertex and several loops, at least as many as the periodicity p of the net. Of course no two loops can have the same voltage but no other restriction is

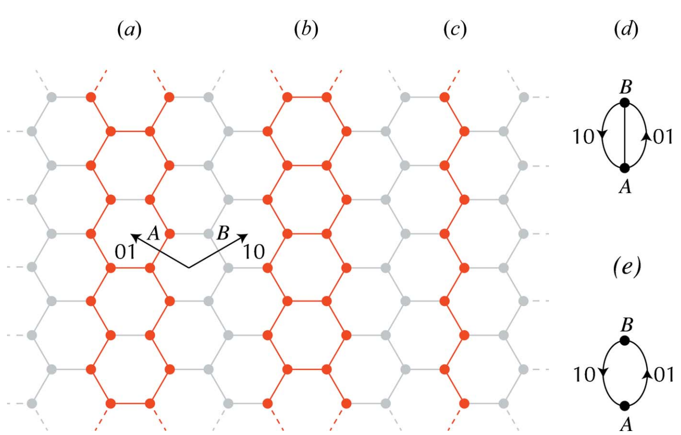

Figure 4
The honeycomb net **hcb** and three 1-periodic subgraphs (in red) that are (*a*) geodesically incomplete, (*b*) geodesically complete but not minimal and (*c*) a fibre along direction 11, together with the labelled quotient graphs of (*d*) **hcb** and (*e*) the fibre along 11.

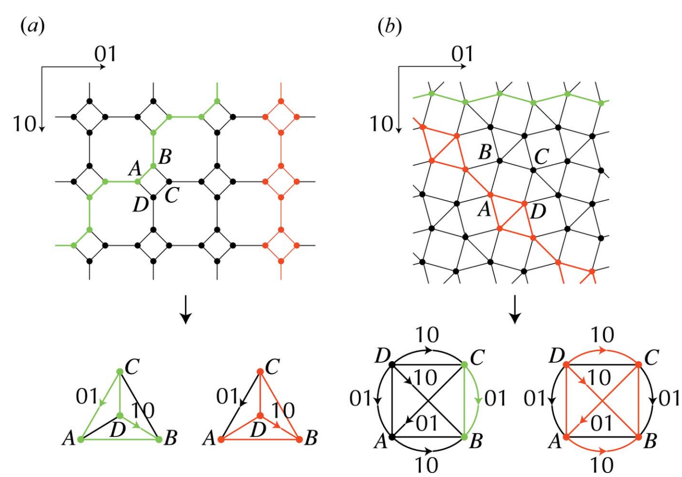

Figure 5
Some geodesic fibres in the 2-periodic nets (*a*) 4.8^2 and (*b*) $4.3.4.3^2$ and their labelled quotient graphs; the same colour is used for the fibre and its quotient, represented as a labelled subgraph of the labelled quotient graph of the net.

imposed. Two examples of 2-periodic lattice nets are presented in Fig. 6. We derive here a fundamental property of these nets.

Theorem 4.1. Lattice nets are crystallographic nets.

Proof. Let (N, T) be a p-periodic lattice net. Because of the restriction on voltages in N/T the quotient graph of any geodesic fibre of N must be a loop. From property (i) we know that there are at least p loops with independent voltages corresponding to geodesic fibres. Let $S \leq T$ be the subgroup generated by this set of voltages: the index of S in T is clearly finite. Suppose now that there exists a non-trivial bounded automorphism $b \notin T$ mapping vertex U to $t(U)$. Then $f = t^{-1}b$ is a non-trivial bounded automorphism with U as a fixed vertex. By property (iii) all geodesic fibres traversing U are also fixed, hence the whole sublattice $S(U)$ is fixed by f. According to Corollary 2.1 any periodic barycentric representation of (N, T), which is also *a fortiori* a periodic barycentric representation of (N, S) – with fixed vertex-lattice $[U]_S = S(U)$ – should display collisions. But this cannot occur since N contains a single vertex-lattice $T(U)$, hence $b = t$. □

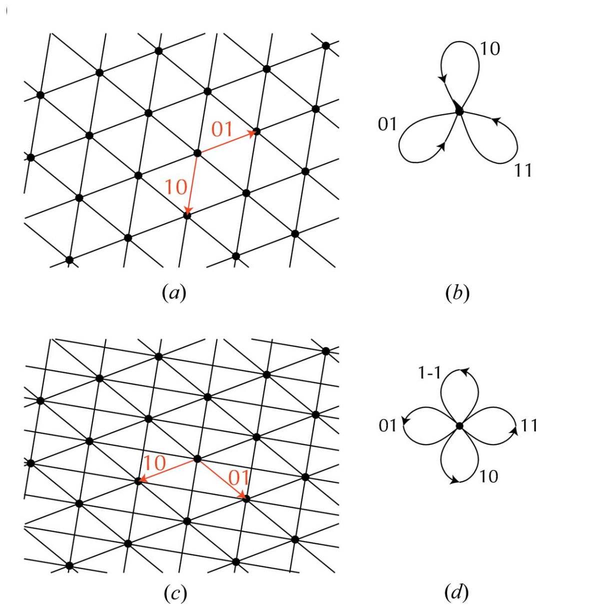

Figure 6
Two lattice nets and their labelled quotient graphs: (*a*) the net **hxl** with (*b*) quotient graph $\mathcal{B}_3$ and (*c*) a net with (*d*) quotient graph $\mathcal{B}_4$.

5. Fibre networks

Different methods may be used to construct a $(p - 1)$-periodic net from a set of geodesic fibres in a p-periodic net, depending on the initial choice of the vertex set. The following definition seems to be more adequate for our purposes. For a p-periodic net (N, T), we consider its fibres along some direction $\langle t \rangle \in T$.

Definition 5.1. The vertex set of a fibre network consists of the family of all geodesic fibres along $\langle t \rangle$ that are equivalent under the bounded automorphism group $B(N)$.

Let then V_t be the vertex set of a fibre network according to Definition 5.1. We may define the distance $d(F_1, F_2)$ in the net N between two geodesic fibres F_1 and F_2 belonging to V_t as the length of a shortest path joining the two fibres. Given any positive integer n, we define the graph $\mathfrak{F}_n$ on V_t such that two fibres F_1 and F_2 in V_t are adjacent in $\mathfrak{F}_n$ whenever $d(F_1, F_2) \leq n$. The fibre network is defined as the graph $\mathfrak{F} = \mathfrak{F}_c$ such that c is the smallest value of n yielding a connected graph. Note that the periodicity of the net N ensures the existence of c.

A few simple examples will illustrate the concept. Fig. 7 shows the **nbo** net and its labelled quotient graph $\mathcal{K}_3^{(2)}$. Note that we use the RCSR three-letter symbols to designate net topologies (O'Keeffe *et al.*, 2008). Geodesic fibres along direction 100 are infinite paths, indicated in green and red for clarity, although they are all equivalent by translation. In agreement with property (ii), the quotient graph F/S of these fibres is the 2-cycle of net voltage 100, marked in orange in $\mathcal{K}_3^{(2)}$. It can be seen in the figure that the minimum distance between two fibres is 2 and that each fibre has eight neighbours at this distance, leading to the 2-periodic fibre net shown as a projection of **nbo**. Because there is only one fibre up to translation, the fibre net is a lattice net and being regular of degree 8 it admits the bouquet $\mathcal{B}_4$ as its quotient graph. Since edges of the fibre net correspond to 2-paths in **nbo** *between* two fibres, we get the voltages over the loops of $\mathcal{B}_4$ as the projection in T/S of the net voltages over 2-paths (or 2-cycles) in $\mathcal{K}_3^{(2)}$ linking vertices in F/S and running along edges that *do not* belong to this quotient, marked in black in Fig. 7. For instance, the 2-cycle B–C–B with net voltage 001 + 111 = 112 originates the loop with voltage 12 and the 2-path B–C–A with net voltage 001 + 000 = 001 originates the loop with voltage 01. In this case, the following intuitive two-step procedure may also be used to get the labelled quotient graph of the fibre

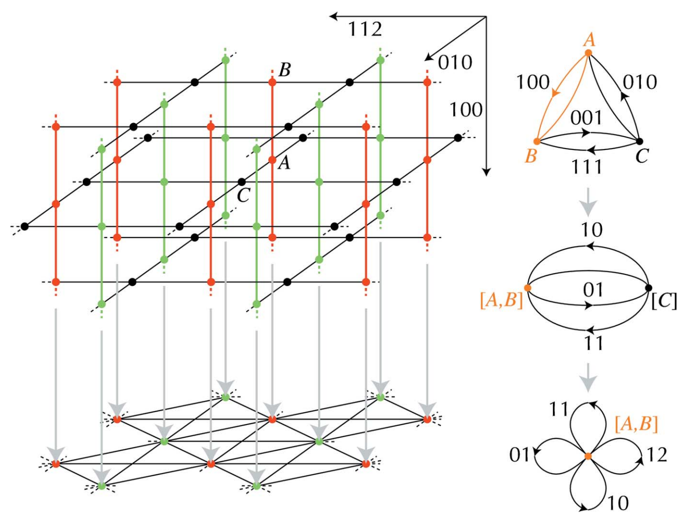

Figure 7
A representation of the **nbo** net (quotient graph: $\mathcal{K}_3^{(2)}$ on the top right) with geodesic fibres along direction 100 drawn along the vertical of the page; the coloured (green and red) fibres of the 3-periodic **nbo** net form the vertex set of a regular 2-periodic net of degree 8 shown as a projection (quotient graph: $\mathcal{B}_4$ on the bottom right). The three labelled vertices *A, B* and *C* mark the origin in the net of the respective vertex-lattices.

network from that of **nbo**. First, the subgraph F/S is condensed into a single (orange) vertex, yielding the new graph $\mathcal{K}_2^{(4)}$, and voltages are substituted by their projection in the quotient group T/S: here we just drop the first coordinate from the triplet label vector of every edge. Note that, in parallel, vertex C must be re-written as $[C]$ and represents a whole class of vertices that are equivalent by the translation 100. In the last step only the fibre vertex is kept and loops are inserted, one for each 2-cycle of $\mathcal{K}_2^{(4)}$ since each 2-cycle is derived from a 2-path in $\mathcal{K}_3^{(2)}$. The voltage over the loop is equal to the net voltage over the respective 2-cycle with the condition that no two loops can have the same, or opposite voltages.

Fig. 8 shows the **pts** net, with quotient graph $\mathcal{C}_4^{(2)}$ shown on the left in Fig. 9. In contrast with **nbo**, geodesic fibres are not degenerated to infinite paths. For clarity, Fig. 10 displays a single fibre of **pts** along direction 100 and its labelled quotient graph, also marked in orange as a subgraph of $\mathcal{C}_4^{(2)}$ in Fig. 9. As in **nbo**, every fibre is also at distance 2 from eight equivalent fibres, and both fibre nets of **nbo** and **pts** along direction 100 are isomorphic. Notice that the isomorphism is already apparent after the first reduction step of the respective labelled quotient graphs. Indeed, the same graph $\mathcal{K}_2^{(4)}$ is obtained in both cases and voltages become equal if the new lattice basis $(1\bar{1}, \bar{1}0)$ is used for **pts**.

We consider finally the NC net associated with sphere packing 4/4/o19 (Sowa, 2012; Moreira de Oliveira Jr & Eon, 2013) drawn in Fig. 11 as a pseudo-barycentric representation (colliding vertices have been circled by pairs). Here geodesic fibres along 001 are infinite paths and there are two translationally non-equivalent fibres with labelled quotient graph marked in orange in Fig. 12. Each fibre is at distance 2 from the colliding fibre and at distance 3 from eight fibres divided into four colliding pairs. Hence $\mathfrak{F}_2$ is not connected; the fibre

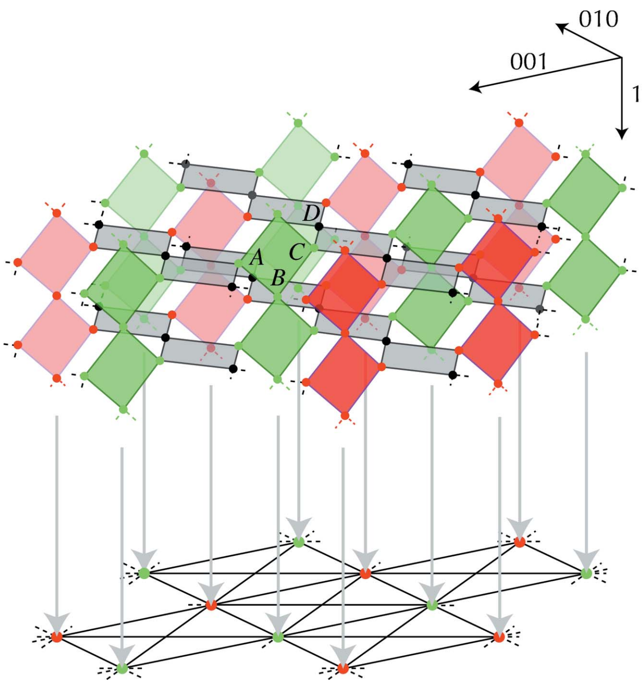

Figure 8
A representation of the **pts** net with fibres along direction 100 drawn along the vertical of the page; the coloured (green and red) fibres of the 3-periodic net form the vertex set of a regular 2-periodic net of degree 8 shown as a projection. For the sake of clarity, fibres are represented as vertex-sharing chains of coloured squares with darker hues for fibres at the front. Fibres along 010, shaded in grey, form a similar pattern cross-linking fibres along 100.

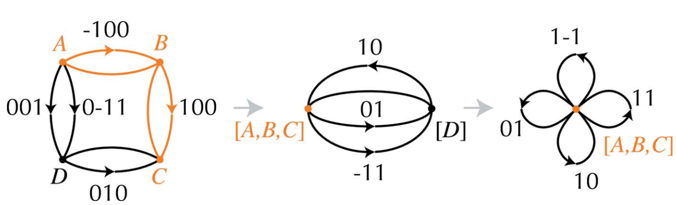

Figure 9
Reduction steps from (*left*) the labelled quotient graph of the **pts** net ($\mathcal{C}_4^{(2)}$) with the labelled quotient graph of the geodesic fibre along 100 marked in orange to (*right*) the labelled quotient graph of the respective fibre network ($\mathcal{B}_4$).

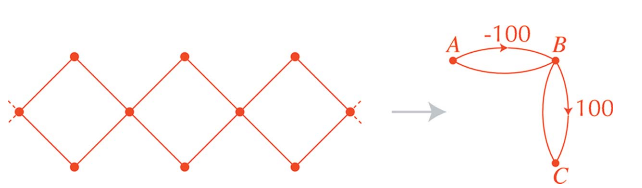

Figure 10
A geodesic fibre of the **pts** net along 100 and its labelled quotient graph.

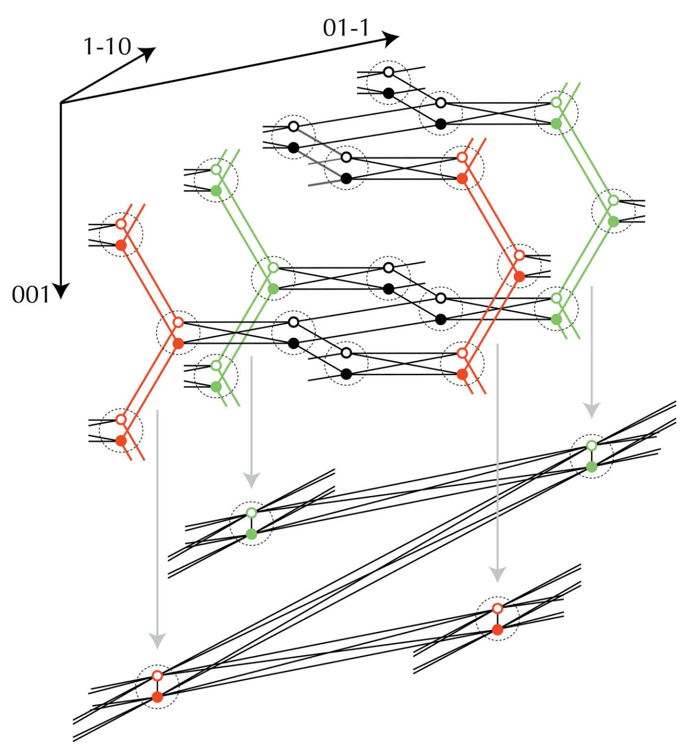

Figure 11
A representation of the underlying net to sphere packing 4/4/o19 with geodesic fibres along direction 001 – drawn along the vertical of the page; the coloured (green and red) fibres of the 3-periodic net form the vertex set of a regular 2-periodic net of degree 9 shown as a projection.

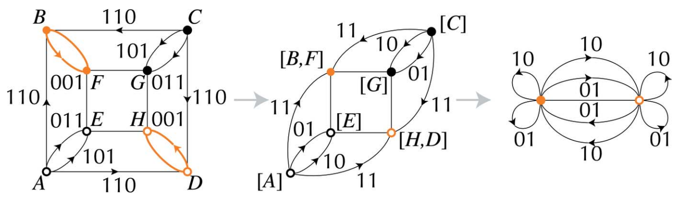

Figure 12
Reduction steps from (*left*) the labelled quotient graph of the underlying net to sphere packing 4/4/*o*19 indicating in orange the labelled quotient graphs of the two colliding geodesic fibres along direction 001 to (*right*) the labelled quotient graph of the respective fibre network.

network is $\mathfrak{F} = \mathfrak{F}_3$ shown as a projection in Fig. 11 and is regular of degree 9.

We now return to general properties of fibre networks. By construction $\mathfrak{F}$ is periodic with periodicity $(p-1)$ but it is not necessarily a periodic net since 3-connectivity may not hold. Let $\varphi \in B(N)$ be a bounded automorphism of N. For any pair of geodesic fibres F_1 and F_2 along $\langle t \rangle$, we know that their images $\varphi(F_1)$ and $\varphi(F_2)$ are geodesic fibres along the same direction $\langle t \rangle$ and moreover $d(\varphi(F_1), \varphi(F_2)) = d(F_1, F_2)$. Hence φ induces an automorphism, say φ^*, in $\mathfrak{F}$. It is clear that φ^* is also a bounded automorphism and that the mapping $\varphi \longmapsto \varphi^*$ is a group homomorphism between Aut(N) and Aut($\mathfrak{F}$). Hence, if φ is of finite order then φ^* is also of finite order. But the converse is not true since any translation along t in N is mapped to $t^* = 1$, the identity in Aut($\mathfrak{F}$). We will now apply the concept of fibre network to prove the stability of the subset of automorphisms of finite order in $B(N)$ under composition. The argument is by induction on the periodicity p of the net. The initial step (1-periodic nets) is proven in the next section. The inductive step is proven in §7.

6. Bounded automorphisms of finite order in 1-periodic networks

An automorphism in a periodic network, considered as a vertex permutation, may be written in cycle form. For automorphisms of finite order n, the cycle decomposition only contains cycles of length at most n. In contrast, bounded automorphisms of infinite order only contain infinite cycles. Suppose, by way of contradiction, that there is one finite cycle of length m in the decomposition of an automorphism φ of infinite order. Then φ^m, another bounded automorphism, has fixed vertices; according to property (iv) of geodesic fibres φ^m has finite order, in contradiction with the hypothesis.

Lemma 6.1. A bounded automorphism of infinite order in a 1-periodic network (N, T) decomposes into finitely many infinite cycles.

Proof. We suppose that the unit cell of the network has been lifted from a spanning tree of the quotient graph N/T. Let then φ be a bounded automorphism of infinite order with norm $|\varphi|$ and let n be the maximum number of cells separating two vertices at distance $|\varphi|$ in N. For the sake of simplicity, we will define a supercell associated with the translation subgroup

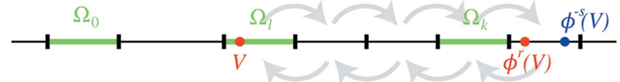

Figure 13
Backward and forward images $\varphi^{-s}(V)$ and $\varphi^{r}(V)$ in the same supercell of a 1-periodic network.

T^n and index the new cells Ω_i for $i \in \mathbb{Z}$ according to the respective translation $t^{ni} \in T = \langle t \rangle$ such that $\Omega_i = t^{ni}(\Omega_0)$ for some origin cell Ω_0. According to these choices, given two vertices A and B in Ω_i and Ω_j, respectively, $d(A, B) > |\varphi|$ whenever $|j - i| > 1$. Call z the order of the supercell (*i.e.*, the number of vertices in the supercell): the vertices in Ω_0 give rise to at most z infinite cycles in the cycle decomposition of φ. Suppose now that not all vertices in supercells Ω_i for $i > 0$ belong to these cycles and take one of them, say $V \in \Omega_l$, at the shortest distance from Ω_0. (The following argument may easily be adapted for $i < 0$.) Then, all the images $\varphi^m(V)$ and $\varphi^{-m}(V)$ for $m > 0$ belong to supercells Ω_i for $i \geq l$. More specifically, for any supercell Ω_k $(k > l)$ there is a value of $r > 0$ such that $\varphi^m(V) \in \Omega_i$ with $i > k$ for all $m \geq r$ (see Fig. 13). The same property holds clearly for φ^{-1}. Let us call D the diameter of the supercell (maximum distance between two vertices in the supercell). According to the definition of the supercell, there is an image $\varphi^{-s}(V)$ $(s > 0)$ in the same supercell containing $\varphi^r(V)$, hence at a distance less than D from this vertex. Thus $d(\varphi^{r+s}(V), V) = d(\varphi^r(V), \varphi^{-s}(V)) < D$, a contradiction since k can be chosen large enough to impose $d(\Omega_k, \Omega_l) > D$. □

Remark 6.1. It is worth observing that the argument developed in the above proof has wider significance. Indeed it implies that the images $\varphi^r(V)$ $(r \in \mathbb{Z})$ of any vertex V by a bounded automorphism φ of infinite order are distributed over the whole network in the sense that every supercell contains at least one image $\varphi^r(V)$ for some r. In the first place the proof showed that it is not possible to find an integer k such that $\varphi^r(V)$ belongs to the supercell $\Omega_{i(r)}$ with $i(r) > k$ for all r nor, equivalently, is it possible to find an integer l such that $\varphi^r(V)$ belongs to the supercell $\Omega_{i(r)}$ with $i(r) < l$ for all r. Hence, for any integer i there are images $\varphi^m(V)$ and $\varphi^n(V)$ such that $\varphi^m(V) \in \Omega_{i(m)}$ and $\varphi^n(V) \in \Omega_{i(n)}$ with $i(m) < i < i(n)$. The definition of the supercell implies then that Ω_i contains at least one image $\varphi^r(V)$ for r in the range between m and n.

We analyse now some properties of permutations in infinite sets. Because any finite cycle may be written as a product of transpositions, we first analyse the product $(U, V)p$ of one or two cyclic permutations p by a transposition (U, V). Our convention is that permutations apply from the right to the left. This product is schematically shown in Fig. 14 where each cyclic permutation is drawn as a directed cycle and the single transposition (U, V) has been represented as a 2-cycle (a double edge): all these cycles have counterclockwise orientation.

Two different situations may arise but in every case ingoing edges are exchanged at U and V as a result of the composition.

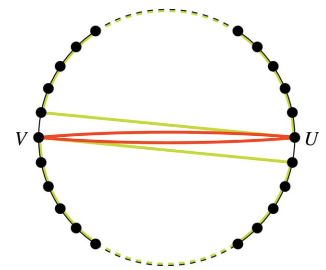

Figure 14
Schematic representation of the product of (black line) one cyclic permutation or (green line) two cyclic permutations by a transposition (red line); all cycles have counterclockwise orientation. Dashed lines indicate that the corresponding cycle may be finite or infinite.

If both vertices U and V belong to the support of the same cycle, it is easily seen in Fig. 14 that the cyclic permutation (in black) followed by the transposition (in red) decomposes into two cyclic permutations (in green). If the initial cycle is infinite, the product decomposes into a finite cycle and an infinite one. The support of the finite permutation contains all the vertices between U and V, including the first vertex U and excluding the last vertex V. The whole operation may then be understood as an excision of the interval $[U, V[$ from the support of the initial cycle with subsequent closure of both cycles.

If, on the other hand, U and V belong to the support of two disjoint cyclic permutations, at least one of them of finite support, the converse clearly occurs. That is, both cycles open at U and V and the finite cycle is inserted into the infinite one. If, however, both cycles are infinite then the product still contains two infinite cycles which have exchanged the semi-infinite parts starting at vertices U and V. We have thus proved the following.

Lemma 6.2. The composition of a finite product of infinite cycles by a transposition may create a finite cycle by excision but it does not change the number of infinite cycles.

For the proof of the next result, we will need a kind of pointwise convergence for permutations in infinite sets. We will say that a sequence p_i, $i \in \mathbb{N}$ of permutations admits the limit p if, for every element U, one can find an integer n such that $p_i(U) = p(U)$ whenever $i > n$.

Lemma 6.3. In a 1-periodic network N, the product of two bounded automorphisms of finite and infinite order, respectively, cannot be an automorphism of finite order. As a consequence, the set of bounded automorphisms of finite order is stable under composition.

Before exposing the proof, it might be useful to offer an informal overview of the main lines of its reasoning. We consider the composition $f\varphi$ of two bounded automorphisms: φ of infinite order and f of finite order, both written in their cycle form. Observe that Lemma 6.2 already ensures the result when f has finite support; the difficulty arises when f has infinite support. The strategy of the proof is to build the product $f\varphi$ as the limit of a sequence $p_m\varphi$ where p_m is a permutation of finite support, which has the same action as f on a finite ball centred at the origin. This is possible because f is a product of finite disjoint cycles C_m which may be applied in any order; they may then be applied by order of proximity to the origin. The argument is based upon the observation that, as the ball is growing at step m and its radius is already larger than the norm $|f| + |\varphi|$, the action of the next cycle C_m is restricted to vertices localized at one of the boundaries of the ball. Notice that convergence is already attained for vertices inside the ball. After composition of p_{m-1} with C_m, infinite cycles in p_m still cross the ball at intervals less than $|f| + |\varphi|$. This property remains up to the limit p. Elimination of the infinite cycles on convergence could only occur if finite cycles C_m acted on both boundaries of the ball, forcing infinite cycles to pull over the ball. An explicit example of the mechanism of convergence to bounded or unbounded permutations in a similar situation is discussed in Appendix A.

Proof. As a bounded automorphism of infinite order, $\varphi = \prod_k \Gamma_k$ contains finitely many infinite disjoint cycles Γ_k. Being of finite order, $f = \prod_{i\in\mathbb{N}} C_i$ contains countably many finite disjoint cycles C_i, which can be decomposed into a finite product of transpositions. Let us define the vertex permutations $p_m = \prod_{i\leq m} C_i\varphi$, or recursively $p_{m+1} = C_{m+1}p_m$ with $p_0 = C_0\varphi$; clearly the sequence p_m converges to the product $f\varphi$ since the cycles C_i are disjoint. Because the product of disjoint cycles commutes, we may choose the sequence C_i so that the support of the permutation $S_m = \prod_{i\leq m} C_i$ covers a ball of radius increasing with m around the origin cell Ω_0 of the network. By repeated application of Lemma 6.2, we see that left multiplication of φ by $\prod_{i\leq m} C_i$ does not change the number of infinite cycles. Notice that every permutation p_m is bounded ($|p_m| \leq |f| + |\varphi|$) but needs not be an automorphism, only the limit $p = f\varphi$ needs to be one. However, the conclusion reached in Remark 6.1 also holds for these infinite cycles because, far from the ball, on both sides, these cycles are identical to those in φ. Hence, the origin cell Ω_0 contains at least one vertex in the support of infinite cycles in the decomposition of p_m at every step m. Clearly, if the limit p were of finite order, every vertex in Ω_0 would belong to a finite cycle for sufficiently large values of m. This shows that the product $f\varphi$ cannot be of finite order, or equivalently that it is not possible to find two bounded automorphisms f and g of finite order such that the product $f^{-1}g = \varphi$ has infinite order. □

7. Bounded automorphisms of finite order in *p*-periodic networks

We still need a preparatory result before generalizing Lemma 6.3 to networks of an arbitrary periodicity.

Lemma 7.1. If the orbit by some bounded automorphism φ of every geodesic fibre in a p-periodic net N is finite, then φ has finite order.

Proof. Call D the diameter of the quotient graph N/T. An arbitrary vertex V in N is at a distance less than D from any fibre in a set of p geodesic fibres along independent directions acting as a set of axes (but not necessarily intersecting) for a frame of reference. Since the orbits of these p fibres are finite, the orbit of V is also finite, showing that some power of φ fixes V. Hence, according to property (iv), φ has finite order. □

Theorem 7.1. The set $F(N)$ of bounded automorphisms of finite order in a p-periodic network (N, T) is stable under composition.

Proof. The proof is by induction on the periodicity of the network. Lemma 6.3 shows that the result is true for 1-periodic networks. Suppose that it is true for periodicity up to p. Let then (N, T) be a $(p+1)$-periodic network and suppose that $\varphi_1, \varphi_2 \in F(N)$ are such that $\varphi_1\varphi_2 \notin F(N)$. According to Lemma 7.1, one can find at least one geodesic fibre F in N whose orbit by the product $\varphi_1\varphi_2$ is infinite. Consider then the corresponding p-periodic fibre network $\mathfrak{F}$ and the images of these automorphisms by the mapping $*$: $\mathrm{Aut}(N) \to \mathrm{Aut}(\mathfrak{F})$. We see that $\varphi_1^*, \varphi_2^* \in F(\mathfrak{F})$, because the mapping $*$ is a group homomorphism but $\varphi_1^*\varphi_2^* = (\varphi_1\varphi_2)^* \notin F(\mathfrak{F})$, in contradiction with the induction hypothesis. □

8. Fundamental theorems for non-crystallographic nets

Let us analyse successively the different kinds of NC nets; it will be seen that every NC net possesses non-trivial bounded automorphisms of finite order.

8.1. NC nets with fixed vertices

We consider first non-crystallographic nets admitting bounded automorphisms with fixed vertices. According to property (iv), such automorphisms have finite order. Let us call $F(N) \subseteq B(N)$ the set of bounded automorphisms of N with finite order. Theorem 7.1 shows that $F(N)$ is stable under composition; it is then easily verified that $F(N)$ is a normal subgroup of $\mathrm{Aut}(N)$.

Theorem 8.1. The orbits by $F(N)$ of the vertex set define a periodic system of imprimitivity with finite blocks for the group of bounded automorphisms.

Proof. We know from Bhattacharjee *et al.* (1998) that the orbits of a normal subgroup form a system of imprimitivity. Suppose that a block Δ contains two translationally equivalent vertices U and $t(U)$. Then there is some $f \in F(N)$ such that $f(U) = t(U)$, or $f^{-1}t(U) = U$, so that $f^{-1}t = g \in F(N)$ (because it has a fixed vertex) and hence $t = fg \in F(N)$ [because of the stability of $F(N)$], which is clearly a contradiction. Therefore, blocks may contain at most one vertex per vertex-lattice and are finite. Let now Δ be the orbit of U under $F(N)$; since $F(N)$ is a normal subgroup, we have $tF(N)t^{-1} = F(N)$, from which we see that $t(\Delta)$ is the orbit of $t(U)$. □

Let σ be the respective partition of the vertex set into blocks of imprimitivity for $B(N)$. It is clear that the setwise stabilizer $B(N)_\sigma$ of σ is a subgroup of $F(N)$, since blocks are finite. By definition $F(N) \leq B(N)_\sigma$, hence $B(N)_\sigma = F(N)$ is transitive on every block. According to Moreira de Oliveira Jr & Eon (2013) it is possible to assign voltages to the labelled quotient graph of the net in such a way that it displays an equivoltage partition. Moreover, any barycentric representation of the net will show vertex collisions.

8.2. NC nets with fixed edges

Suppose now that some NC net admits non-trivial bounded automorphisms that fix edges but no non-trivial bounded automorphism fixing vertices. This implies that the respective automorphism, say φ, exchanges the two end-vertices of the fixed edge and that $\varphi^2 = 1$ is the identity of $\mathrm{Aut}(N)$. The *ladder* in Fig. 2(a) provides an example of such automorphisms. We may deal with these nets by inserting a vertex into every fixed edge. The new net possesses bounded automorphisms of order 2 fixing the added vertices but with no fixed edges. Hence there is a periodic system of imprimitivity for $B(N)$ with finite blocks such that (i) the two vertices linked to the added one belong to the same block and (ii) the added vertex belongs to another block. If we delete the inserted vertex and link again the two end-vertices, we get a non-trivial periodic system of imprimitivity with finite blocks for the initial net where the two end-vertices of the fixed edge still

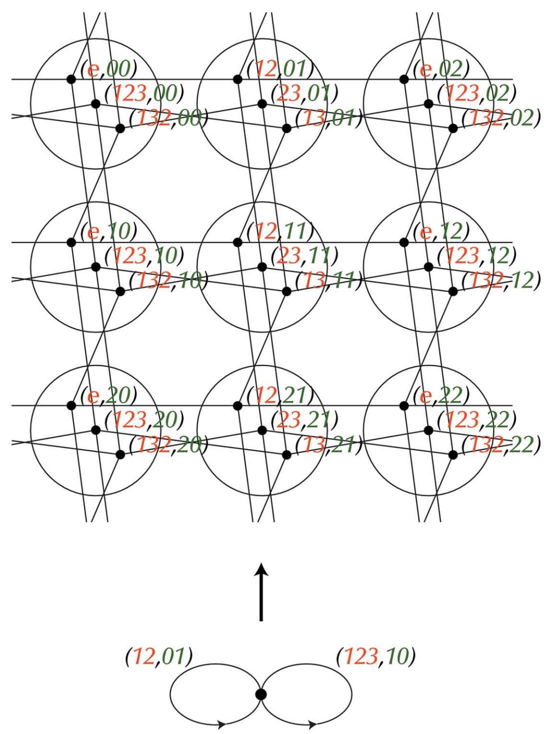

Figure 15
Derivation of a 2-periodic NC net from the bouquet $\mathcal{B}_2$ with voltages $((1, 2); 01)$ and $((1, 2, 3); 10)$ with permutations $(1, 2)$ and $(1, 2, 3)$ in $\mathcal{S}_3$ and translations 01 and 10 in $\mathbb{Z}^2$. For the sake of clarity, permutations (i, j, k) are noted ijk. Vertices of the net may be labelled as (p, t) in the direct product $\mathcal{S}_3 \times \mathbb{Z}^2$. According to the definition (Gross & Tucker, 2001) the voltage graph implies that vertex (p, t) is linked to vertices $(p(1, 2), t + 01)$ and $(p(1, 2, 3), t + 10)$. The resulting net is isomorphic to that shown in Fig. 1(b).

belong to the same block. This shows that the conclusions of the previous subsection concerning the existence of an equivoltage partition of the labelled quotient graph and that of collisions in any barycentric representation of the net also hold in this case.

8.3. NC nets with freely acting bounded automorphisms

It was shown in Moreira de Oliveira Jr & Eon (2011) that any NC net N with freely acting, non-abelian, bounded automorphism group $B(N)$ may be derived from a finite graph by assigning voltages (h_i, t_i) in the direct product $\mathcal{H} \times \mathbb{Z}^n$ to its edges, where $\mathcal{H}$ is some finite non-abelian permutation group. A simple example is displayed in Fig. 15.

The group of bounded automorphisms $B(N)$ of such nets is isomorphic to the subdirect product of $\mathcal{H}$ and $\mathbb{Z}^n$ generated by the voltages (h_i, t_i). Suppose now that the two permutations h_i and h_j do not commute in $\mathcal{H}$; the commutator $[(h_i, t_i), (h_j, t_j)] = ([h_i, h_j], 0)$ corresponds to a non-trivial bounded automorphism of finite order. Hence the subgroup $F(N) < B(N)$ is not trivial, which shows that such nets also possess a periodic system of imprimitivity with finite blocks.

Suppose finally that an abelian group of bounded automorphisms $B(N)$ acts freely on the net. Then we can take the (finite) quotient $N/B(N)$ and assign voltages to the respective edges in $B(N)$, which shows that $B(N)$ is finitely generated. According to the fundamental theorem about finitely generated abelian groups, $B(N)$ is a direct product of cyclic groups (Kargapolov & Merzljakov, 1979). The case $B(N)$ free abelian corresponds to a crystallographic net. If $B(N)$ is abelian but not free, then there is some automorphism of finite order and we may again conclude as above. We have thus shown the following fundamental result.

Theorem 8.2. Any non-crystallographic net admits a labelled quotient graph with an equivoltage partition. Any periodic barycentric representation of the net shows vertex collisions; vertices that are equivalent under bounded automorphisms of finite order project on the same Euclidean point.

The converse of the last affirmation is not true and this may lead to a classification scheme on NC nets. Given an NC net with system of imprimitivity σ, it may happen that the periodic net $N/\sigma = N_1$ is also non-crystallographic, with system of imprimitivity σ_1. Let us denote $N_2 = N_1/\sigma_1$ and inductively $N_{k+1} = N_k/\sigma_k$. The order of the labelled quotient graph N_{k+1}/T, which is the quotient of N_k/T by the equivoltage partition, is strictly smaller than that of the latter if the partition is not the trivial one. Since lattice nets (with quotients of order 1) are crystallographic nets, the procedure must end with a crystallographic net after a finite number of steps. We shall say that an NC net is of *type k* if the quotient N_k is a crystallographic net. Examples of NC nets of types 2 and 3 are given in §§9 and 10, respectively.

8.4. Structure of the bounded automorphism group

From the previous analysis, it appears that the quotient group $B(N)/F(N)$ is generally a subgroup of the group of bounded automorphisms of the quotient network N/σ. Let $b \in B(N)$ and consider the class of automorphisms $bF(N) \in B(N)/F(N)$. Since only automorphisms in the identity class $F(N)$ can have finite order, the quotient $B(N)/F(N)$ has no non-trivial automorphism of finite order. According to the conclusions reached in §§8.1 to 8.3, $B(N)/F(N)$ is then a free abelian group S, and hence isomorphic to a translation subgroup of the periodic net N/σ. This yields our last theorem.

Theorem 8.3. The group of bounded automorphisms $B(N)$ of the NC net (N, T) is isomorphic to the semi-direct product of the translation group T and the subgroup $F(N)$ of bounded automorphisms of finite order: $B(N) \simeq T \ltimes F(N)$.

Proof. Clearly $t_1F(N) \neq t_2F(N)$ if $t_1, t_2 \in T$ are distinct, hence $T \leq S$. Suppose there is some $b \in B(N)$ which is mapped to a translation $s \notin T$ of N/σ. Because S and T have the same rank, T has finite index in S and there is some power $s^n \in T$, as a translation of N/σ. This shows that one may use b to obtain an extension of T as a translation group of N. If we suppose that T is a maximal translation group of (N, T) we have a contradiction, hence $S = T$. □

This last result also provides a simple interpretation to bounded automorphisms $b \in B(N)$. Because they can be written as a product $b = ft$ of a translation $t \in T$ followed by an automorphism of finite order $f \in F(N)$, one can say that any bounded automorphism approximates some translation. Blocks of imprimitivity being finite, there is an upper bound, say $|F|$, to the norm of any automorphism in $F(N)$, hence $d(b(V), t(V)) \leq |F|$ for any vertex V in the net. Moreover, the bounded automorphism ft has the same effect as the translation t on the barycentric representation of the net.

9. A simple NC net based on nbo

Fig. 16(*a*) shows a pseudo-barycentric representation of an NC net derived from **nbo**. Its labelled quotient graph, Fig. 16(*b*), displays an equivoltage partition with only one non-trivial cell: the two respective vertices have been enclosed within a small orange circle. The corresponding vertices in each block of the net can exchange quite independently from the remaining blocks. This gives $F(N) \simeq (\mathcal{S}_2)^{\mathbb{Z}^3}$ so that the bounded automorphism group is isomorphic to the wreath product $\mathbb{Z}^3$ Wr $\mathcal{S}_2$. Taking the quotient of the labelled quotient graph in Fig. 16(*b*) by the equivoltage partition yields the labelled quotient graph in Fig. 16(*c*). This graph also shows an equivoltage partition with three cells formed by blue, green and red pairs of vertices, respectively. The quotient by this partition is the labelled quotient graph of the **nbo** net shown in Fig. 16(*d*). Of course, being a stable net, **nbo** is a crystallographic net, hence the NC net is of type 2. It may also be seen on the figure that the NC net and **nbo** have the same barycentric representation.

After the two circled vertices have been identified in the quotient N/σ, a single new bounded automorphism turns into being. The respective labelled quotient graph, Fig. 16(*c*), has in fact an automorphism of order 2 which preserves the voltages over all cycles; the corresponding automorphism of the net commutes with the translation group so that $F(N/\sigma) \simeq \mathcal{S}_2$ and $B(N/\sigma) \simeq \mathcal{S}_2 \times \mathbb{Z}^3$. It should be noted that the labelled quotient graph N/T, Fig. 16(*b*), also displays an automorphism which preserves the voltages over its cycles, indicating the existence of a bounded automorphism of the net that respects its periodic structure. This automorphism (also of order 2) exchanges simultaneously every pair of colliding vertices. Thus, automorphisms of the labelled quotient graph do not give enough information to characterize the group $B(N)$. Analysis of the correlation groups shows that permutations inside cells of the equivoltage partition have no correlation in the case of the net N but are completely correlated in the case of its quotient N/σ.

10. A simple NC net based on pcu

As a further example, Fig. 17(*a*) shows a representation of an NC net derived from **pcu**. Its labelled quotient graph, Fig. 17(*b*), displays an equivoltage partition with only one non-trivial cell containing the two vertices A and B. The corresponding vertices in each block of the net cannot exchange independently from the remaining blocks in directions 010 and 001. Exchanging two vertices in one block demands exchanging the two vertices in every block in the same plane orthogonal to direction 100. This gives $F(N) \simeq (\mathcal{S}_2)^{\mathbb{Z}}$ so that the bounded automorphism group is isomorphic to the product $(\mathbb{Z} \text{ Wr } \mathcal{S}_2) \times \mathbb{Z}^2$. Taking the quotient of the labelled quotient graph in Fig. 17(*b*) by the equivoltage partition yields the labelled quotient graph in Fig. 17(*c*). This graph also shows an equivoltage partition with two cells, the quotient of which, shown in Fig. 17(*d*), admits the bouquet $\mathcal{B}_3$, *i.e.* the labelled quotient graph of **pcu** as final quotient. Of course **pcu** is a crystallographic net, hence the NC net is of type 3. It may also be seen that the NC net and **pcu** have the same barycentric representation with the four vertices colliding in the unit cell. We emphasize that, although vertices A and C are not equivalent under the automorphism group of the net, indeed they even have different degrees, they collide in the barycentric representation of the net.

Clearly the reverse procedure may be used to generate NC nets, that is: starting from the labelled quotient graph of a crystallographic net, one may replicate several times every vertex together with the incident edges, including their labelled vectors, to get a quotient graph with an equivoltage partition.

11. Final considerations

We have shown that NC nets are unstable nets, *i.e.* their barycentric representations display vertex collisions. This means that, conversely, stable periodic nets are crystallographic nets.

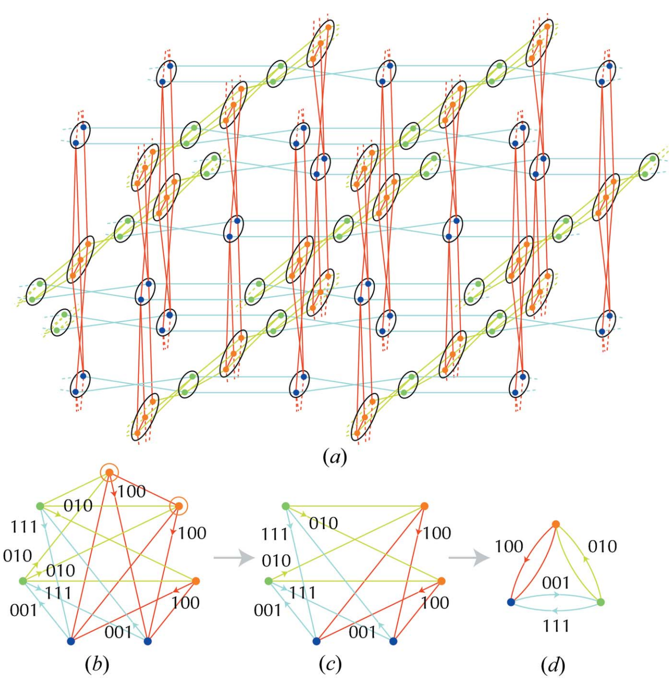

Figure 16
(*a*) A representation of an NC net derived from **nbo** and the sequence of labelled quotient graphs showing equivoltage partitions starting from (*b*) the labelled quotient graph of the net to (*d*) the labelled quotient graph of **nbo** (see text).

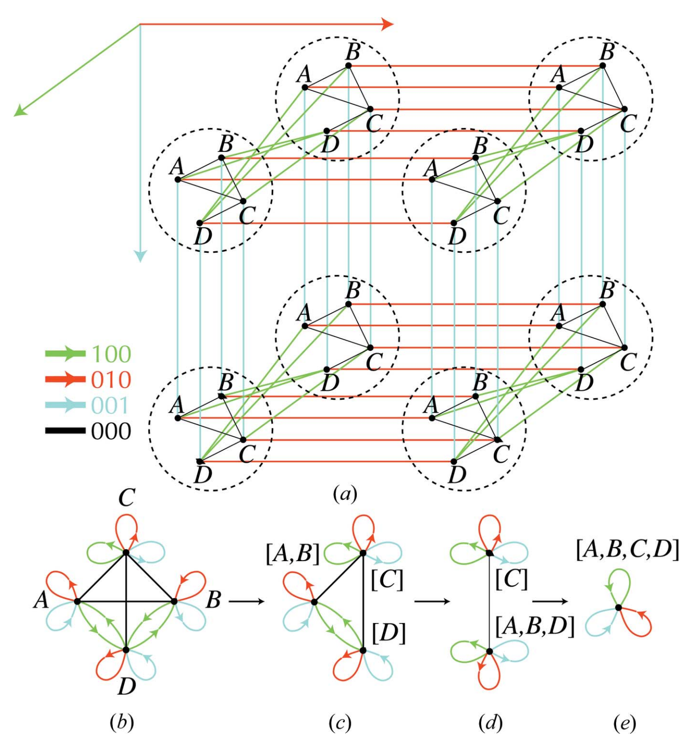

Figure 17
(*a*) A representation of an NC net derived from **pcu** and the sequence of labelled quotient graphs showing equivoltage partitions starting from (*b*) the labelled quotient graph of the net to (*e*) the labelled quotient graph of **pcu** (see text).

A further question comes from the existence of unstable crystallographic nets. NC nets have a system of finite blocks of imprimitivity for the group of bounded automorphisms and, as a consequence, their labelled quotient graph admits an equivoltage partition. It was shown in Moriera de Oliveira Jr & Eon (2013) that the labelled quotient graph of some crystallographic nets also admits an equitable partition, which enforces vertex collisions for the barycentric representation of the net. However, trivial correlation groups between different cells forbid the existence of bounded automorphisms of finite order. Hence, even if some unstable net admits an equivoltage partition, the analysis of this partition is necessary before classifying it as NC.

We end with a note of caution concerning the converse question: some labelled quotient graphs do not admit equivoltage partitions but nonetheless correspond to unstable crystallographic nets (Delgado-Friedrichs *et al.*, 2013). The case of quotient graphs with a bridge is well known [see, for example, in Eon (1999) the 2-periodic minimal net derived from the dumbbell graph] but general conditions for instability have not yet been recognized.

APPENDIX *A*
Convergence of permutations in $\mathbb{Z}$

Let us say that a permutation p of the set of integers $\mathbb{Z}$ is bounded if there is an integer $|p|$ such that $|p(i) - i| < |p|$ for any $i \in \mathbb{Z}$. Define the translation $t : i \mapsto i + 1$ of infinite order and two permutations of finite order, $f = \prod_i (2i, 2i + 1)$ and $g = \prod_i (i, -i)$ for $i \in \mathbb{Z}$; t and f are bounded, g is not. The first product, $ft : 2i - 1 \mapsto 2i + 1$, contains a single infinite cycle translating all odd numbers and fixes every even number. It is bounded but of infinite order. The second product $gt = \prod_i (-i, i - 1)$ contains infinitely many 2-cycles: it is of finite order but it is not bounded.

In the first case, we may define $p_m = \prod_{|i| \leq m} (2i, 2i + 1)t$, which fixes even numbers in the interval $-2m \leq i \leq 2m$, maps i to $i + 1$ for $|i| > 2m$ and $2i - 1$ to $2i + 1$ for $|i| < 2m$. Clearly p_m converges to ft, every p_m contains a single infinite cycle, as well as the limit.

In the second case, we define $p_m = \prod_{i \leq m} (i, -i)t$, which contains m 2-cycles $(-i, i - 1)$ for $i < m$ and contains one infinite cycle that maps i to $i + 1$ for $i < -m - 1$ and $i \geq m$, but maps $-m - 1$ to m. In other words, the infinite cycle is equal to t in the two semi-infinite parts $i < -m - 1$ and $i \geq m$ but pulls directly from $-m - 1$ to m. Although every p_m contains a single infinite cycle, its limit gt has none.

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